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Abstract

The stochastic multi-path Traveling Salesman Problem aims at finding the expected minimum-cost Hamiltonian tour in a network characterized by the presence of different paths between each pair of nodes, given that a random travel cost with an unknown probability distribution is associated with each of these paths. When the path travel costs are independent and identically distributed, previous works have proved that this problem can be deterministically approximated. Such an approximation has been shown to be of good quality in terms of estimation of an optimal solution with respect to consolidated approaches such as Stochastic Programming with recourse, totally overcoming the computational burden of solving enormous programs exploded by the scenarios considered. Nevertheless, in real settings, the hypothesis about the independence among the path travel costs is far to be reasonable. It is well-known, in fact, that traffic congestion affects travel costs and creates dependence among them. Then the independence assumption for the travel costs does not hold anymore. In this paper we show that the independence assumption can be relaxed and a deterministic approximation of the stochastic multi-path Traveling Salesman Problem by assuming just asymptotically independent travel costs can be derived. We also show that this deterministic approximation has strong operational implications because it allows to deal with realistic traffic models. Computational tests on extensive sets of random and realistic instances show very good efficiency and accuracy of the deterministic approximation.

Keywords: Traveling Salesman Problem, stochastic travel costs, asymptotic independence, deterministic approximation.

1 Introduction

Due to its theoretical interest and wide applicability, the Traveling Salesman Problem (TSP) is undoubtedly one of the most studied problems in combinatorial optimization. Several logistics and routing problems, as well as other combinatorial problems (such as *job scheduling*), can be modeled as a TSP or contain the TSP as a critical sub-problem. In the classical TSP version, travel costs are deterministically known a priori and associated to arcs representing a unique way to go from one node to another (in general, a shortest path between the two nodes).

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However, in real routing applications uncertainty deriving by many factors (accidents, traffic congestion, bad weather conditions, etc.) may strongly affect travel costs. Furthermore, the decision maker can often choose among different travel paths between the same pair of nodes, as it commonly happens in multi-modal transportation networks. The stochastic multi-path Traveling Salesman Problem (*smpTSP*), introduced in Tadei, Perboli, and Perfetti (2014), addresses these two features simultaneously. The authors showed that, despite of the increase in the complexity of the problem, the savings obtained by explicitly incorporating the stochasticity into a multi-path TSP strongly justify the effort. Unfortunately, the time needed to solve the *smpTSP* increases exponentially with the problem size and the number of the considered scenarios. Hence, a deterministic approximation has been developed to solve real-life size instances. In Maggioni, Perboli, and Tadei (2014), the same approximation is applied to a City Logistics real application in which routing instances from the literature have been extended to incorporate real data collected from a sensor network. This approximation, as many other methods existing in the literature for similar problems (cfr. Section 2), assumes that travel cost oscillations are independent and identically distributed (i.i.d.). It is well-known that in real settings network travel costs are far from being independent, in particular under network congestion, where the time delay is propagated through arcs.

For this reason, in this paper we investigate a *smpTSP* variant called stochastic multi-path Travelling Salesman Problem with dependent travel costs ($smpTSP_{dc}$), in which travel cost oscillations are still assumed to be identically distributed but just asymptotically independent.

The contribution of this work is manifold. First, to the best of our knowledge, we address a TSP problem taking into account both multi-path networks and dependence of the travel cost oscillations for the first time. As already said, not the independence but just the asymptotic independence is assumed for that random oscillations. Second, we show that this assumption is not too restrictive in real applications and actually allow to incorporate into the problem realistic traffic models based on the well-known *Wardrop's traffic equilibrium* principle (Wardrop 1952). Third, we formally prove that a deterministic approximation along the lines of the one already proposed in Tadei, Perboli, and Perfetti (2014) can be still derived by using the asymptotic theory of extreme values (Galambos 1978).

We would like to stress the fact that this new theoretical result is not actually problem-dependent and therefore can be generalized and applied to solve other similar optimization problems under uncertainty. Finally, our deterministic approximation provides a powerful and quite accurate decision support tool to deal with the $smpTSP_{dc}$. Its quality and efficiency are assessed through extensive sets of computational experiments, in which both random networks and realistic traffic models are considered.

The remainder of this paper is organized as follows. In Section 2, we briefly review the literature available on the stochastic TSP and we point out some common assumptions. In Section 3, we present the mathematical model of the problem. In Section 4, we discuss the existing links between the Wardrop's traffic equilibrium principle and the asymptotic independence of the travel costs of a network. In Section 5, we prove our main results enabling us to develop a deterministic approximation of the $smpTSP_{dc}$. Finally, in Section 6 we present the performance of the proposed approximation by means of several numerical examples. Section 7 gives conclusions of the work and sketch some possible future investigations.

2 Literature review

The Traveling Salesman Problem (TSP) is one of the most well-known $\mathcal{NP}$-complete combinatorial optimization problems. It appears in many practical applications, either directly or as a sub-problem. Even if many excellent books devoted to the TSP have been published in the past (Lawler et al. 1985, Reinelt 1994, Gutin and Punnen 2002), the problem continues to arise considerable interest among researchers. Consequently, several different generalizations or versions of the problem have appeared in the literature. Vehicle Routing (Toth and Vigo 2014), Orienteering (Vansteenwegen, Souffriau, and Oudheusden 2011), and Travelling Purchaser (Manerba, Mansini, and Riera-Ledesma 2017) problems all belong to this broad class. Most of these routing problems have been also addressed in their stochastic counterpart (see, e.g., Kenyon and Morton 2003, Campbell, Gendreau, and Thomas 2011, Beraldi et al. 2017).

Of particular importance the work of Kirkpatrick and Toulouse (1985) that introduces the stochastic version of the TSP. A TSP model is said to be stochastic if at least one of its components is assumed to be a random variable. The most traditional approach adopted to solve the stochastic TSP is to assume that all the random variables considered in the problem are i.i.d. with a given distribution. For example, in Carraway, Morin, and Moskowit (1989), Kao (1978), Sniedovich (1981), Huang et al. (2018), the authors study the TSP with independent and normally distributed arc costs. However, the restrictive assumptions of those problems are not sufficient to ensure that deterministic methods work in the stochastic setting, as it is the case for the Shortest Path Problem under an exponential probability distribution of the costs (see Eiger, Mirchandani, and Soroush 1985).

In other papers (e.g. Wästlund 2010, Mezard and Parisi 1986), the authors approach the stochastic TSP by means of Statistical Mechanics tools such as the *mean field* approximation and the *replica and cavity* methods. In all previous works, the arc costs are considered to be i.i.d. uniformly or exponentially. Other important studies consider variants of the stochastic TSP problem. For example, in Campbell and Thomas (2008) the authors present two recourse problems and one chance constrained model formalizing the stochastic TSP where there is a deadline associated with each node.

As one may expect, the results derived in all the aforementioned papers are strongly related to the properties of the underlying distribution. Instead, in many real applications, travel costs are determined by very complex mechanisms and thus a precise derivation of the distribution that describes the variations of the arc costs is a difficult, or even impossible, task. Nevertheless, in the literature, there are papers where the authors overcome this problem. For example, in Toriello, William, and Poremba (2013), the authors present and study a dynamic stochastic model of the TSP in which the realizations of the random costs vector connecting a single node to the others is known only when the salesman is about to leave that particular node. They show that, regardless of the distribution, if the costs are assumed to be independent with known expected values and supports, the problem can be formulated as a dynamic programming problem solvable by approximating it through a Linear Programming (LP) model. Another paper that considers a wide class of distributions is Tadei, Perboli, and Perfetti (2014). In this work, the authors prove that, if the random costs are i.i.d. according to a probability distribution belonging to the Gumbel distribution domain of attraction, it is possible to derive an asymptotic approximation of the expected minimum Hamiltonian tour by using the extreme value theory.

Despite the fact that the above papers propose approaches that allow to consider different types of distributions, the i.i.d. assumption on the random arc costs makes them not applicable in many real situations. Recently, a large number of papers have studied both spatial and temporal correlation among travel times in real-life road networks (Fan, Kalaba, and Moore 2005, Samaranayake, Blandin, and Bayen 2012, Chen et al. 2014). They found that real networks have strongly dependent arc costs. This happens, for example, when vehicles are prone to delays due to rush hours, road works, accidents, or generally speaking when the traffic is congested. Unfortunately, the vehicle routing literature is still lacking in considering both stochastic and dependent costs. Only in Letchford and Nasiri (2015), the authors study a Steiner TSP with stochastic correlated costs and find a Pareto frontier through integer programming techniques.

In this work, we consider stochastic and dependent costs and show that just an asymptotic independence among random travel costs is required to derive a good deterministic approximation and justify the use of realistic traffic models.

The mathematical description of flow principles in real traffic networks is an active and demanding field of research. Since the topic is not central in our discussion, we just recall one of the most known results in the field, i.e. the *first Wardrop's traffic equilibrium principle* (Wardrop 1952, Wardrop and Whitehead 1952). That principle states that at the equilibrium no single driver can unilaterally reduce his/her travel cost by shifting to another route. In other words, the traffic tends to be distributed such that all alternative paths between two nodes show the same cost. Since its introduction in the context of road traffic research, transportation planners have been developed Wardrop's equilibrium-based models to predict commuters decisions in real-life networks. Some models have been and are still used today to evaluate alternative future scenarios and to plan future actions on the networks. Other models, instead, describe how the traffic flow increases with respect to the traffic conditions, such as the *U.S. Bureau of Public Roads* (BPR) function (U.S. Department of Commerce, Bureau of Public Roads 1964). We are going to consider this last model in order to show how random dependencies affect the traffic network.

3 Problem definition and mathematical formulation

Let us consider:

- $G = (\mathcal{I}, \mathcal{E})$: directed complete graph;
- $\mathcal{I}$: node set;
- $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{I}, i \neq j\}$: arc set;
- $\mathcal{P}_{ij}$: path set for arc $(i, j) \in \mathcal{E}$;
- ω: random variable belonging to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible outcomes, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability function;
- $C_{ij}^p(\omega)$: stochastic travel cost for arc $(i, j) \in \mathcal{E}$ on path $p \in \mathcal{P}_{ij}$;
- c_{ij}^p: deterministic travel cost for arc $(i, j) \in \mathcal{E}$ on path $p \in \mathcal{P}_{ij}$;
- $\Theta_{ij}^p(\omega)$: random oscillation of the deterministic travel cost c_{ij}^p for arc $(i, j) \in \mathcal{E}$ on path $p \in \mathcal{P}_{ij}$.

We assume

$$C_{ij}^p(\omega) := c_{ij}^p + \Theta_{ij}^p(\omega), \quad (i, j) \in \mathcal{E}, p \in \mathcal{P}_{ij}. \tag{1}$$

The stochastic multi-path Traveling Salesman Problem (*smpTSP*) aims at finding the expected minimum-cost Hamiltonian tour in G and deciding which path must be used to travel between each pair of nodes in that tour.

In the following, we propose a two-stage Stochastic Programming (SP) formulation with recourse for the *smpTSP*. Let us consider a binary variable y_{ij}, $(i, j) \in \mathcal{E}$, taking value 1 if node j is visited directly after node i, and 0 otherwise, and a binary variable x_{ij}^p, $(i, j) \in \mathcal{E}$ and $p \in \mathcal{P}_{ij}$, taking value 1 if path p is selected for travelling across arc (i, j), and 0 otherwise. The variables y_{ij} are the first-stage variables deciding which arcs compose the Hamiltonian tour. Instead, the x_{ij}^p are the second-stage recourse variables modeling the decision on which path to select among each pair of nodes (e.g., which transport mode must be used) in the Hamiltonian tour decided at the first stage.

The *smpTSP* is defined as follows

$$\min_{x,y} \quad \mathbb{E}_{\mathbb{P}}\left[\sum_{(i,j)\in\mathcal{E}} \sum_{p\in\mathcal{P}_{ij}} x_{ij}^p(\omega)\, C_{ij}^p(\omega)\right] \tag{2}$$

subject to

$$\sum_{(i,j)\in\mathcal{E}} y_{ij} = 1, \quad i \in \mathcal{I} \tag{3}$$

$$\sum_{(j,i)\in\mathcal{E}} y_{ij} = 1, \quad j \in \mathcal{I} \tag{4}$$

$$\sum_{i\in U} \sum_{j\notin U} y_{ij} \geq 1, \quad U \neq \emptyset, U \subset \mathcal{I} \tag{5}$$

$$\sum_{p\in\mathcal{P}_{ij}} x_{ij}^p(\omega) = y_{ij}, \quad (i,j) \in \mathcal{E} \tag{6}$$

$$x_{ij}^p(\omega) \in \{0,1\}, \quad (i,j) \in \mathcal{E}, p \in \mathcal{P}_{ij} \tag{7}$$

$$y_{ij} \in \{0,1\}, \quad (i,j) \in \mathcal{E}. \tag{8}$$

The objective function (2) minimizes the expected total travel cost. Constraints (3) and (4) are the assignment constraints ensuring that each node is visited once and only once, while connectivity constraints (5) prevent the formation of sub-tours in the solution. Constraints (6) link together variables x_{ij}^p and y_{ij}. In particular, when arc (i, j) is not selected by the first stage ($y_{ij} = 0$), no path belonging to that arc (i, j) can be used. On the contrary, when arc (i, j) is selected by the first stage ($y_{ij} = 1$), then one and only one path must be selected for that arc. Finally, (7) and (8) are binary conditions on the variables.

Please note that the above formulation is slightly different from the one proposed in Tadei, Perboli, and Perfetti (2014), which included a non-linear objective function. This new formulation, instead, leads to an Integer Linear Program formulation of the deterministic equivalent problem (see Section 3.1).

3.1 A Deterministic Equivalent Problem (DEP) formulation

The stochastic model (2)-(8) is nearly impossible to solve because of the difficulty to calculate the expected value in the objective function as a multi-dimensional integral, which cannot be solved analytically. A common SP approach to overcome this problem (see, e.g., Wallace and Ziemba 2005) is to discretize the probability distribution of the random variables, by creating a finite number of possible realizations (called *scenarios*), and then to approximate the stochastic model with a deterministic one named indeed Deterministic Equivalent Problem (DEP).

Hence, in the following, we consider a set $\mathcal{S}$ of possible scenarios. Each scenario $s \in \mathcal{S}$, occurring with a probability π^s, is associated with a random cost oscillation Θ_{ij}^{ps} for each arc $(i,j) \in \mathcal{E}$ and path $p \in \mathcal{P}_{ij}$. Because π^s is a probability, we have $\sum_{s\in\mathcal{S}} \pi^s = 1$. The DEP of the *smpTSP* can be stated as follows

$$\min_{x,y} \sum_{s\in\mathcal{S}} \pi^s \sum_{(i,j)\in\mathcal{E}} \sum_{p\in\mathcal{P}_{ij}} x_{ij}^{ps}\ C_{ij}^{ps} \tag{9}$$

subject to

$$\sum_{(i,j)\in\mathcal{E}} y_{ij} = 1, \quad i \in \mathcal{I} \tag{10}$$

$$\sum_{(j,i)\in\mathcal{E}} y_{ij} = 1, \quad j \in \mathcal{I} \tag{11}$$

$$\sum_{i\in U} \sum_{j\notin U} y_{ij} \geq 1, \quad U \neq \emptyset,\ U \subset \mathcal{I} \tag{12}$$

$$\sum_{p\in\mathcal{P}_{ij}} x_{ij}^{ps} = y_{ij}, \quad (i,j) \in \mathcal{E}, s \in \mathcal{S} \tag{13}$$

$$x_{ij}^{ps} \in \{0,1\}, \ \ (i,j) \in \mathcal{E}, p \in \mathcal{P}_{ij}, s \in \mathcal{S} \tag{14}$$

$$y_{ij} \in \{0,1\}, \ \ (i,j) \in \mathcal{E} \tag{15}$$

where $C_{ij}^{ps} := c_{ij}^{p} + \Theta_{ij}^{ps}$ for each arc $(i,j) \in \mathcal{E}$, path $p \in \mathcal{P}_{ij}$, and scenario $s \in \mathcal{S}$.

It is worthwhile noticing that model (9)–(15), although deterministic, has some strong drawbacks. First, it explodes in complexity with the size of $\mathcal{S}$ and, therefore, finding an optimal solution by considering a reasonable number of scenarios can be computationally intractable. Second, in order to create the scenario set, it is necessary to have a precise knowledge about the distribution of all the random variables involved.

Our approach will overcome both the above drawbacks. In fact, the complexity of the deterministic model resulting from our approximation presented in Section 5.2 is not affected by the number of scenarios, and the knowledge of the random variables distribution is not necessary.

3.2 The *smpTSP* with dependent travel costs ($smpTSP_{dc}$)

As highlighted in Section 2, *smpTSP* has been always studied assuming that the travel costs are independent and identically distributed (i.i.d). In this paper, instead, we specifically address a generalization of the *smpTSP* in which the random variables $\Theta_{ij}^{p}(\omega)$

- have an unknown joint probability distribution;
- are just *asymptotically independent* (more precisely, in this particular case, we are interested in an asymptotic independence *on the left tail* of the random cost distributions).

Definition 1. *Let X_1 and X_2 be two random variables. They are said to be asymptotically independent on their left tail if*

$$\lim_{r \to -\infty} \big(\mathbb{P}[X_1 \le r \mid X_2 \le r] - \mathbb{P}[X_1 \le r] \big) = 0. \tag{16}$$

In other words, assuming two random variables to be asymptotically independent on their left tail means that the probability to have a variation of one variable in its left tail is not influenced by the knowledge of a variation on the left tail of the other variable.

Relaxing the strong independence assumption allows us to address the traffic congestion effects in real networks, where travel costs are far to be independent.

4 Wardrop's first principle and asymptotic independence of the travel costs

In this section we discuss how requiring the random oscillations to be asymptotically independent on their left tail (see Def. 1) is not at all a restrictive condition to exhaustively model the stochastic behavior of the network traffic due to congestion, justifying the overall approach presented in this paper and the assumptions done in Section 3.2.

Our argumentation is done by using the well-known concept of *user equilibrium* based on the Wardrop's first principle of route choice (Wardrop 1952, Wardrop and Whitehead 1952) and thus proving that travel costs are actually highly correlated but still asymptotically independent on their left tail. We recall that the Wardrop's principle says: "*The traffic arranges itself in congested networks such that all used routes between an origin and a destination pair have equal and minimum costs, while all unused routes show greater costs*". This means that, at the equilibrium, each traveler cannot obtain savings in travel costs by choosing a different path. The Wardrop's principle basically derives from the classical Game Theoretic field, and in particular from the Nash's equilibrium (see, e.g., Osborne and Rubinstein 1994).

We start by pointing out the following

Remark 1. *In real applications, the support of random travel cost oscillations is such that the total cost of a path remains non-negative. Hence, for each $(i,j) \in \mathcal{E}$, $p \in \mathcal{P}_{ij}$, the random cost oscillation Θ_{ij}^p is lower-bounded by $-c_{ij}^p$. Thus, we can show the property of asymptotic independence of the oscillations on the left tail by verifying that, for any pair of different paths $p^1, p^2 \in \mathcal{P}_{ij}$ of any arc $(i,j) \in \mathcal{E}$, the following condition holds*

$$\lim_{r \to -c_{ij}^{p^1}, t \to -c_{ij}^{p^2}} \mathbb{P}[\Theta_{ij}^{p^1} \le r \mid \Theta_{ij}^{p^2} \le t] = 0. \tag{17}$$

Now, let us denote by $\mathcal{P}_{ij}^U \subseteq \mathcal{P}_{ij}$ the subset of paths used at the equilibrium for arc $(i,j) \in \mathcal{E}$. From the Wardrop's first principle and Remark 1, if $p^1, p^2 \in \mathcal{P}_{ij}^U$, then the relative random costs $C_{ij}^{p^1 s}$ and $C_{ij}^{p^2 s}$ are equal for each scenario $s \in \mathcal{S}$, and, consequently

$$\Theta_{ij}^{p^1 s} - \Theta_{ij}^{p^2 s} = c_{ij}^{p^2} - c_{ij}^{p^1}, \quad (i,j) \in \mathcal{E}. \tag{18}$$

Equation (18) shows the high correlation which exists among the random cost oscillations.

We now investigate the behavior of such oscillations on the left tail of their joint distribution. As commonly done in transportation engineering studies, we assume that the travel cost on a given path increases when the flow of traffic increases on the same path (see, e.g., U.S. Department of Commerce, Bureau of Public Roads 1964, where the cost of each path is evaluated by the *Bureau of Public Road* function). More precisely, we assume that there exists an increasing function H_{ij}^{ps} such that

$$C_{ij}^{ps} = H_{ij}^{ps}(Q_{ij}^{ps}), \quad (i,j) \in \mathcal{E},\ p \in \mathcal{P}_{ij}^{U},\ s \in \mathcal{S} \tag{19}$$

in which Q_{ij}^{ps} denotes the actual flow of traffic on path $p \in \mathcal{P}_{ij}^{U}$ under scenario $s \in \mathcal{S}$. Please note that the considered function H_{ij}^{ps} also depends on the particular conditions of a specific scenario s, such as bad weather or accidents. We now consider path $p^2 \in \mathcal{P}_{ij}^{U}$ and suppose the event $\Theta_{ij}^{p^2 s} \leq t$ occurs with t very closed to $-c_{ij}^{p^2}$. Path p^2 will show a cost slightly different than 0 and, due to the Wardrop's first principle, it holds that

$$C_{ij}^{p^2 s} \leq C_{ij}^{p^1 s}, \quad p^1 \in \mathcal{P}_{ij}^{U}. \tag{20}$$

This means that new users entering the network and willing to move from node i to node j would prefer to use path p^2 instead of other paths. Consequently, because of (19) the flow of traffic on p^2 will increase and that on the other paths will decrease. This process will continue until a new equilibrium is reached. However, during all the above process, it holds that

$$C_{ij}^{p^1 s} > C_{ij}^{p^2 s} > 0, \quad (i,j) \in \mathcal{E},\ p^1 \in \mathcal{P}_{ij}^{U},\ s \in \mathcal{S}. \tag{21}$$

Thus, the random cost oscillation $\Theta_{ij}^{p^1 s}$ for path $p^1 \in \mathcal{P}_{ij}^{U}$ is not expected to assume a value close to its lower bound $-c_{ij}^{p^1}$. This proves that the travel cost oscillations are asymptotically independent on the left tail of their distribution.

5 Deterministic approximation of the stochastic problem

To develop the deterministic approximation presented in this section, we consider the $smpTSP_{dc}$ problem as a discrete choice model where the decision maker will select the best alternative among a finite set of mutually exclusive ones, i.e., the best path to move from node i to node j. The approximation works in two main steps. The first in which it is possible to derive how the costs of the best alternatives are asymptotically distributed, and the second where an estimator for the travel cost oscillations can be analytically determined. This approach has been already used in other application domains such as location, routing, loading, and packing problems (Perboli, Tadei, and Baldi 2012, Tadei et al. 2012, Perboli, Tadei, and Gobbato 2014). However, the independence of the stochastic variables has never been relaxed before.

In order to set our approximation, we adopt an optimistic view (i.e., guided by the objective function of the $smpTSP_{dc}$) and relax the problem by assuming that we can choose among all scenarios the one that minimizes the random travel cost oscillations. More precisely, we define $\tilde{\Theta}_{ij}^{p}$ as the minimum random travel cost oscillations Θ_{ij}^{ps} among all scenarios $s \in \mathcal{S}$, i.e.

$$\tilde{\Theta}_{ij}^{p} := \min_{s \in \mathcal{S}} \Theta_{ij}^{ps}, \quad (i,j) \in \mathcal{E},\ p \in \mathcal{P}_{ij}. \tag{22}$$

We also define $F_{ij}^{p}(x)$ as the survival function of $\tilde{\Theta}_{ij}^{p}$, i.e., $F_{ij}^{p}(x) = \mathbb{P}[\tilde{\Theta}_{ij}^{p} > x]$.

Remark 2. *Since $\Theta_{ij}^{p}(\omega)$ are asymptotically independent on their left tail, then also $\tilde{\Theta}_{ij}^{p}$ are asymptotically independent on their left tail.*

Now, it is easy to see that, among all alternative paths from node i to node j, the path showing the lowest cost will be selected in the optimal solution of the $smpTSP_{dc}$. For the sake of simplicity and without loss of generality, we assume such path to be unique. We define $C_{ij}(\Theta)$ as the cost of such optimal path for traveling from node i to node j, i.e.

$$C_{ij}(\Theta) := \min_{p \in \mathcal{P}_{ij}} \left(c_{ij}^p + \tilde{\Theta}_{ij}^p \right), \quad (i,j) \in \mathcal{E} \tag{23}$$

Note that $C_{ij}(\Theta)$ is still a random variable since depending on $\tilde{\Theta}_{ij}^p$. We call its survival function

$$G_{ij}(x) = \mathbb{P}[C_{ij}(\Theta) > x]. \tag{24}$$

Obviously, a variable x_{ij}^p will take value 1 in an optimal solution of the $smpTSP_{dc}$ if and only if p is the optimal path from i to j and, therefore, variables x_{ij}^p can be surrogated by the already existing variables y_{ij}. Hence, because of the linearity of the expected value, problem (2)-(8) becomes

$$\min_y \sum_{(i,j) \in \mathcal{E}} \mathbb{E}_{\mathbb{P}} \left[C_{ij}(\Theta) \right] y_{ij} \tag{25}$$

subject to constraints (3)–(5), and (8).

Unfortunately, the distribution of $C_{ij}(\Theta)$ is unknown because the distribution of $\tilde{\Theta}_{ij}^p$ is unknown. Thus the expected value in (25) is not solvable. We will provide in the following Section an asymptotic approximation of the distribution of $C_{ij}(\Theta)$, or, equivalently, of its survival function $G_{ij}(x)$.

5.1 Asymptotic approximation of $G_{ij}(x)$

Please note that, by subtracting a constant α from all random cost oscillations Θ_{ij}^{ps}, the optimal solution of problem (2)-(8) does not change. In fact, let us denote by

$$f_0 := \mathbb{E}_{\mathbb{P}} \left[\sum_{(i,j) \in \mathcal{E}} \sum_{p \in \mathcal{P}_{ij}} x_{ij}^p(\omega) * \left(c_{ij}^p + \Theta_{ij}^p(\omega) \right) \right]$$

the original objective function, and by

$$f := \mathbb{E}_{\mathbb{P}} \left[\sum_{(i,j) \in \mathcal{E}} \sum_{p \in \mathcal{P}_{ij}} x_{ij}^p(\omega) * \left(c_{ij}^p + \Theta_{ij}^p(\omega) - \alpha \right) \right]$$

the same objective function after the normalization of the cost oscillations. Then, the following condition holds

$$\begin{aligned} f &= f_0 - \alpha \sum_{(i,j) \in \mathcal{E}} \sum_{p \in \mathcal{P}_{ij}} x_{ij}^p(\omega) = \\ &= f_0 - \alpha \sum_{(i,j) \in \mathcal{E}} y_{ij} = \\ &= f_0 - \alpha |\mathcal{E}|. \end{aligned}$$

Hence, we can restate (23) as

$$C_{ij}(\Theta) = \min_{p \in \mathcal{P}_{ij}} (c_{ij}^p + \min_{s \in \mathcal{S}} (\Theta_{ij}^{ps} - \alpha_{|\mathcal{S}|})), \quad (i,j) \in \mathcal{E} \tag{26}$$

where $\alpha_{|\mathcal{S}|}$ is chosen equal to the root of the equation

$$1 - F_{ij}^p(x) = \frac{1}{|\mathcal{S}|}. \tag{27}$$

Theorem 1. *Let us consider any arc $(i,j) \in \mathcal{E}$. If the random cost oscillations Θ_{ij}^p of each path $p \in \mathcal{P}_{ij}$ are asymptotically independent on their left tail, and if*

$$\lim_{|\mathcal{S}| \to +\infty} (F_{ij}^p(x + \alpha_{|\mathcal{S}|}))^{|\mathcal{S}|} = exp(-e^{\beta x}) \text{ for some real number } \beta > 0 \tag{28}$$

then

$$\lim_{|\mathcal{S}| \to +\infty} G_{ij}(x) = \lim_{|\mathcal{S}| \to +\infty} \mathbb{P}[C_{ij}(\Theta) > x] = \lim_{|\mathcal{S}| \to +\infty} \mathbb{P}\left[\min_{p \in \mathcal{P}_{ij}} (c_{ij}^p + \min_{s \in \mathcal{S}} (\Theta_{ij}^{ps} - \alpha_{|\mathcal{S}|})) > x\right] = e^{-A_{ij} e^{\beta x}} \tag{29}$$

where

$$A_{ij} = \sum_{p \in \mathcal{P}_{ij}} e^{-\beta c_{ij}^p}. \tag{30}$$

Proof. Let $P_{ij} = |\mathcal{P}_{ij}|$ and $F_{ij}(x_1, x_2, \ldots, x_{P_{ij}})$ be the unknown joint survival function of all the P_{ij} random oscillations associated to the paths connecting node i to node j under any given scenario $s \in \mathcal{S}$, i.e.

$$F_{ij}(x_1, x_2, \ldots, x_{P_{ij}}) = \mathbb{P}[\bigcap_{p=1,2,\ldots,P_{ij}} \Theta_{ij}^{ps} > x_p]. \tag{31}$$

Using the De Morgan's laws and the property of the probability over the union of a finite number of events, it turns out that

$$\begin{aligned} F_{ij}(x_1, x_2, \ldots, x_{P_{ij}}) &= \mathbb{P}[\bigcap_{p=1,2,\ldots,P_{ij}} \Theta_{ij}^{ps} > x_p] = \\ &= 1 - \mathbb{P}[\bigcup_{p=1,2,\ldots,P_{ij}} \Theta_{ij}^{ps} \le x_p] = \\ &= 1 - \sum_{k=1}^{P_{ij}} (-1)^{k+1} \sum_{\{p^1, p^2, \ldots, p^k\} \in 2^{\mathcal{P}_{ij}}} \mathbb{P}[\Theta_{ij}^{p^1 s} \le x_1, \Theta_{ij}^{p^2 s} \le x_2, \ldots, \Theta_{ij}^{p^k s} \le x_k] \end{aligned} \tag{32}$$

where $2^{\mathcal{P}_{ij}}$ is the power set of $\mathcal{P}_{ij}$, i.e. the set containing all subsets of the set $\mathcal{P}_{ij}$.

Without loss of generality, let us consider two paths $p^1, p^2 \in \mathcal{P}_{ij}$. From Def. 1, it is easy to see that

$$\lim_{\substack{x_1 \to -\infty, \\ x_2 \to -\infty}} \mathbb{P}[\Theta_{ij}^{p^1 s} \le x_1 | \Theta_{ij}^{p^2 s} \le x_2] = 0, \quad s \in \mathcal{S} \tag{33}$$

or, equivalently

$$\lim_{\substack{x_1 \to -\infty, \\ x_2 \to -\infty}} \frac{\mathbb{P}[\Theta_{ij}^{p^1 s} \le x_1, \Theta_{ij}^{p^2 s} \le x_2]}{\mathbb{P}[\Theta_{ij}^{p^2 s} \le x_2]} = 0, \quad s \in \mathcal{S} \tag{34}$$

Note that Eq. (34) can be generalized as follows

$$\lim_{\substack{x_1 \to -\infty, \\ x_2 \to -\infty}} \frac{\mathbb{P}[\Theta_{ij}^{p^1 s} \le x_1, \Theta_{ij}^{p^2 s} \le x_2]}{\mathbb{P}[\Theta_{ij}^{p^2 s} \le \min(x_1, x_2)]} = 0, \quad s \in \mathcal{S} \tag{35}$$

In fact, when $x_1 \to -\infty$ and $x_2 \to -\infty$, it holds that

$$0 \le \frac{\mathbb{P}[\Theta_{ij}^{p^1 s} \le x_1, \Theta_{ij}^{p^2 s} \le x_2]}{\mathbb{P}[\Theta_{ij}^{p^2 s} \le \min(x_1, x_2)]} \le \frac{\mathbb{P}[\Theta_{ij}^{p^1 s} \le \min(x_1, x_2), \Theta_{ij}^{p^2 s} \le \min(x_1, x_2)]}{\mathbb{P}[\Theta_{ij}^{p^2 s} \le \min(x_1, x_2)]} \to 0. \tag{36}$$

The limit in (35) has the following very important interpretation: when both x_1 and x_2 tend to $-\infty$, one has that $\mathbb{P}[\Theta_{ij}^{p^1 s} \le x_1, \Theta_{ij}^{p^2 s} \le x_2] \to 0$. Hence in (32), if $x_1 \to -\infty, x_2 \to -\infty, \ldots, x_{P_{ij}} \to -\infty$ all probabilities including the intersection of 2 or more events are negligible. Thus, when $x_1 \to -\infty, x_2 \to -\infty, \ldots, x_{P_{ij}} \to -\infty$,

$$F_{ij}(x_1, x_2, \ldots, x_{P_{ij}}) \to 1 \ - \sum_{p=1,2,\ldots,P_{ij}} \mathbb{P}[\Theta_{ij}^{ps} \le x_p] \tag{37}$$

under any scenario $s \in \mathcal{S}$.

Due to (26), $\mathbb{P}[C_{ij}(\theta) > x]$ can be written as a function of $|S|$ as follows

$$\begin{aligned}
\mathbb{P}[C_{ij}(\theta) > x] &= \mathbb{P}[\min_{p\in\mathcal{P}_{ij}} (c_{ij}^p + \min_{s\in\mathcal{S}}(\Theta_{ij}^{ps} - \alpha_{|\mathcal{S}|})) > x] = \\
&= \mathbb{P}[\bigcap_{p\in\mathcal{P}_{ij}} (c_{ij}^p + \min_{s\in\mathcal{S}}(\Theta_{ij}^{ps} - \alpha_{|\mathcal{S}|})) > x] = \\
&= \mathbb{P}[\bigcap_{p\in\mathcal{P}_{ij}} (\min_{s\in\mathcal{S}}(\Theta_{ij}^{ps} - \alpha_{|\mathcal{S}|})) > x - c_{ij}^p] = \\
&= \mathbb{P}[\bigcap_{p\in\mathcal{P}_{ij}} \bigcap_{s\in\mathcal{S}} (\Theta_{ij}^{ps} - \alpha_{|\mathcal{S}|}) > x - c_{ij}^p] = \\
&= \mathbb{P}[\bigcap_{p\in\mathcal{P}_{ij}} \bigcap_{s\in\mathcal{S}} \Theta_{ij}^{ps} > x - c_{ij}^p + \alpha_{|\mathcal{S}|}] = \\
&= \mathbb{P}[\bigcap_{s\in\mathcal{S}} \bigcap_{p\in\mathcal{P}_{ij}} \Theta_{ij}^{ps} > x - c_{ij}^p + \alpha_{|\mathcal{S}|}] = \\
&= \prod_{s\in\mathcal{S}} \mathbb{P}[\bigcap_{p\in\mathcal{P}_{ij}} \Theta_{ij}^{ps} > x - c_{ij}^p + \alpha_{|\mathcal{S}|}] = \\
&= [F_{ij}(x - c_{ij}^{p^1} + \alpha_{|\mathcal{S}|}, x - c_{ij}^{p^2} + \alpha_{|\mathcal{S}|}, \ldots, x - c_{ij}^{p^{P_{ij}}} + \alpha_{|\mathcal{S}|})]^{|\mathcal{S}|}
\end{aligned} \tag{38}$$

From the assumption (28) it is easy to see that, when $|\mathcal{S}| \to +\infty$

$$F_{ij}^p(x + \alpha_{|\mathcal{S}|}) \to 1, \quad p \in \mathcal{P}_{ij} \tag{39}$$

In fact, if $F_{ij}^p(x + \alpha_{|\mathcal{S}|})$ were bounded by any real number $a < 1$, then $(F_{ij}^p(x + \alpha_{|\mathcal{S}|}))^{|\mathcal{S}|}$ would tend to 0 for any real number x and that would contradict (28).

Using (39) one has that $\lim_{|\mathcal{S}|\to\infty}(x + \alpha_{|\mathcal{S}|}) = -\infty, \ \ x \in \mathbb{R}$. Thus, under any scenario $s \in \mathcal{S}$

$$\lim_{|\mathcal{S}|\to\infty} (x - c_{ij}^p) + \alpha_{|\mathcal{S}|} = -\infty, \quad p \in \mathcal{P}_{ij}, x \in \mathbb{R} \tag{40}$$

Due to (37) and (40), when $|\mathcal{S}| \to +\infty$

$$F_{ij}((x_1 - c_{ij}^{p^1}) + \alpha_{|\mathcal{S}|}, (x_2 - c_{ij}^{p^2}) + \alpha_{|\mathcal{S}|}, \ldots, (x_{P_{ij}} - c_{ij}^{p^{P_{ij}}}) + \alpha_{|\mathcal{S}|}) \to 1 - \sum_{p\in\mathcal{P}_{ij}} \mathbb{P}[\Theta_{ij}^{ps} \le (x_p - c_{ij}^p) + \alpha_{|\mathcal{S}|}] \tag{41}$$

Hence, by using (38) and (41), one gets

$$\lim_{|\mathcal{S}|\to+\infty} \mathbb{P}[C_{ij}(\theta) > x] = \lim_{|\mathcal{S}|\to+\infty} e^{|\mathcal{S}| \log(1 - \sum_{p\in\mathcal{P}_{ij}} \mathbb{P}[\Theta_{ij}^{ps} \le (x - c_{ij}^p) + \alpha_{|\mathcal{S}|}])} \tag{42}$$

Since (40) implies that $\lim_{|\mathcal{S}|\to+\infty} \mathbb{P}[\Theta_{ij}^{ps} \le (x_p - c_{ij}^p) + \alpha_{|\mathcal{S}|}] = 0, p \in \mathcal{P}_{ij}, \ \ s \in \mathcal{S}$, then (42) leads to

$$
\begin{aligned}
\lim_{|\mathcal{S}|\to+\infty} \mathbb{P}[C_{ij}(\theta) > x] &= \lim_{|\mathcal{S}|\to+\infty} e^{-|\mathcal{S}|(\sum_{p\in\mathbb{P}_{ij}} \mathbb{P}[\Theta_{ij}^{ps} \le (x - c_{ij}^{p}) + \alpha_{|\mathcal{S}|}])} = \\
&= \lim_{|\mathcal{S}|\to+\infty} (e^{-(\sum_{p\in\mathbb{P}_{ij}} \mathbb{P}[\Theta_{ij}^{ps} \le (x - c_{ij}^{p}) + \alpha_{|\mathcal{S}|}])})^{|\mathcal{S}|} = \\
&= \lim_{|\mathcal{S}|\to+\infty} \prod_{p\in\mathcal{P}_{ij}} (e^{-\mathbb{P}[\Theta_{ij}^{ps} \le (x - c_{ij}^{p}) + \alpha_{|\mathcal{S}|}]})^{|\mathcal{S}|} = \\
&= \lim_{|\mathcal{S}|\to+\infty} \prod_{p\in\mathcal{P}_{ij}} (1 - \mathbb{P}[\Theta_{ij}^{ps} \le (x - c_{ij}^{p}) + \alpha_{|\mathcal{S}|}]))^{|\mathcal{S}|} = \\
&= \prod_{p\in\mathcal{P}_{ij}} \lim_{|\mathcal{S}|\to+\infty} (F_{ij}^{p}((x - c_{ij}^{p}) + \alpha_{|\mathcal{S}|}))^{|\mathcal{S}|} \qquad (43)
\end{aligned}
$$

Now, due to (28) and (43), it holds that

$$
\lim_{|\mathcal{S}|\to\infty} \mathbb{P}[C_{ij}(\theta) > x] = \prod_{p\in\mathcal{P}_{ij}} \exp(-e^{\beta(x - c_{ij}^{p})}) = e^{-A_{ij}e^{\beta x}}. \qquad (44)
$$

This proves the theorem. □

Basically, Theorem 1 states that, if the unknown probability distribution of the stochastic cost oscillations satisfies assumption (28), then the costs asymptotically converge in distribution to a Gumbel function (double exponential) even if the costs are asymptotically independent. Note that the expression A_{ij} in (30) represents the so-called *accessibility* in the sense of Hansen (1959), which is a measure of "visibility" that the decision maker has for each arc (i, j) on the entire set of its alternative paths $\mathcal{P}_{ij}$. In turn, this accessibility depends on a parameter $\beta > 0$ that must be calibrated (see Section 6.1) and represents the dispersion of the alternatives in the decision making process, i.e. the propensity to choose for an arc among the set of its paths characterized by different random travel times.

5.1.1 Applicability of Theorem 1.

After having presented, proved, and commented Theorem 1, we want to dedicate a brief discussion to highlight the vast applicability of the results obtained by showing the mildness of assumption (28) made on the structure of the distribution of the random cost oscillations.

First, note that assumption (28) can be equivalently rewritten as

$$
\lim_{|\mathcal{S}|\to+\infty} (F_{ij}^{p}(\frac{1}{\beta}x + \alpha_{|\mathcal{S}|}))^{|\mathcal{S}|} = \exp(-e^{x})
$$

and thus, for an accurate calibration of β, it just requires the distribution to belong to the *domain of attraction* of the Gumbel distribution. This domain constitutes a very large family of distributions including very common ones as the Normal, the Gumbel, the Weibull, the Logistic, the Laplace, the Lognormal, and many others (i.e., any distribution of the form $1 - e^{-P(x)}$, where $P(x)$ is a polynomial function). Hence, (28) can be actually considered a very mild assumption.

Second, it is important to notice that (28) is a more general assumption with respect to the one used in Tadei, Perboli, and Perfetti (2014) and in similar approaches already appeared in the literature, where a more restrictive behaviour on the distribution tails was imposed. In particular, the following asymptotic exponential behavior for the left tail of the distribution $F_{ij}^p(x)$ was required:

$$\lim_{y \to -\infty} \frac{1 - F_{ij}^p(x+y)}{1 - F_{ij}^p(y)} = e^{\beta x} \quad \text{for some real number } \beta > 0. \tag{45}$$

We prove in the following Proposition that (28) is a more general assumption than (45).

Proposition 1. *Assumption in* (45) *implies assumption in* (28).

Proof. From (27) it holds that $\lim_{|\mathcal{S}| \to +\infty} \alpha_{|\mathcal{S}|} = -\infty$. From (45) it turns out that

$$\lim_{|\mathcal{S}| \to +\infty} \frac{1 - F_{ij}^p(x + \alpha_{|\mathcal{S}|})}{1 - F_{ij}^p(\alpha_{|\mathcal{S}|})} = e^{\beta x}. \tag{46}$$

By using (27), (46) becomes

$$\lim_{|\mathcal{S}| \to +\infty} \frac{1 - F_{ij}^p(x + \alpha_{|\mathcal{S}|})}{\frac{1}{|\mathcal{S}|}} = e^{\beta x}$$

and, thus

$$\lim_{|\mathcal{S}| \to +\infty} F_{ij}^p(x + \alpha_{|\mathcal{S}|}) = \lim_{|\mathcal{S}| \to +\infty} \left(1 - \frac{e^{\beta x}}{|\mathcal{S}|}\right).$$

Hence

$$\lim_{|\mathcal{S}| \to +\infty} (F_{ij}^p(x + \alpha_{|\mathcal{S}|}))^{|\mathcal{S}|} = \lim_{|\mathcal{S}| \to +\infty} \left(1 - \frac{e^{\beta x}}{|\mathcal{S}|}\right)^{|\mathcal{S}|} = \exp(-e^{\beta x}).$$

□

This means that any distribution satisfying (45) also satisfies our assumption. On the contrary, it is easy to see that some of the already mentioned distributions satisfying (28), e.g. the Normal and the Lognormal, do not show the behavior expressed in (45).

5.2 Deterministic approximation of $smpTSP_{dc}$

It is worthwhile noticing that, given the result of Theorem 1, it is also possible to derive for each arc a Multinomial Logit model for the choice probability of its alternative paths (see Tadei, Perboli, and Manerba 2018). This could lead to a continuous assignment of paths to arcs, and to a possible feasible solution of our $smpTSP_{dc}$ through rounding. However, given the hard feasibility constraints of our problem, we have noticed in preliminary experiments that this rounding leads to quite bad approximated solutions. Therefore, we have decided instead to exploit the knowledge of the asymptotic distribution of the random cost oscillations to calculate their expected value, so to achieve an approximated model for the problem based only on deterministic parameters.

More precisely, if $|\mathcal{S}|$ is large enough, the limit obtained in (44) can be used as the survival function of costs $C_{ij}(\Theta)$ and therefore we can calculate their expected value as follows

$$\mathbf{E}_\Theta[C_{ij}(\Theta)] = \int_{-\infty}^{+\infty} x \; d\mathbb{P}[C_{ij}(\theta) \le x] = -\int_{-\infty}^{+\infty} x \; d\mathbb{P}[C_{ij}(\theta) > x] = \int_{-\infty}^{+\infty} x e^{-A_{ij}e^x} A_{ij} e^x dx \tag{47}$$

After some manipulations, the above expected value becomes

$$\mathbf{E}_\Theta[C_{ij}(\Theta)] \approx -\frac{1}{\beta}(\ln(A_{ij}) + \gamma) \tag{48}$$

where $\gamma = -\int_0^\infty \log(t)e^{-t}dt \approx 0.5772$ is the Euler constant.

Now, by using (48) and disregarding the constant terms, the following deterministic approximation of the stochastic problem (2)-(8) is obtained:

$$\min_{y} \quad -\frac{1}{\beta} \sum_{(i,j)\in\mathcal{E}} y_{ij} \ln(A_{ij}) \tag{49}$$

subject to constraints (3)–(5), and (8).

Note that the deterministic approximation developed allows to reduce the combinatorial structure of the DEP of our $smpTSP_{dc}$ in formulation (9)–(15) to a common TSP, overcoming the complexity deriving by the presence of both multiple paths and multiple scenarios.

6 Computational experiments

In order to assess the performance of the proposed approach, we compare the results obtained by the Deterministic Approximation (DA) proposed in Section 5.2 with those of the Deterministic Equivalent Problem (DEP) formulated in (9)–(15) on a large set of benchmark instances. The DA have been implemented by using *MATLAB* v9.4 and its internal integer solver, whereas the DEP have been solved by using Cplex v12.7.1 and its C++ Concert Technology. In all the experiments, we have considered a discretization of the probability space in 100 scenarios ($|\mathcal{S}| = 100$). This choice enforces both *in-sample* and *out-of-sample* stability of the problem (see Kaut et al. 2007). We run all the experiments on an Intel Core I7 2.5 GHz workstation with 16GB RAM, running *Windows* 10 operating system.

In Section 6.1 we propose an empirical way to calibrate the parameter β, needed to calculate our deterministic approximation. In Section 6.2 we discuss the generation of the instances. In Section 6.3 the computational results are given.

6.1 Calibration of parameter β

As already said, the DA depends on the parameter β that needs to be calibrated. In all the experiments, the calibration of β is done as in Tadei, Perboli, and Perfetti (2014). More precisely, let us consider the standard Gumbel distribution $\exp(-e^{-x})$. If an approximation error of 2‰ is accepted, then $\exp(-e^{-x}) = 1 \iff x = 6.08$ and $\exp(-e^{-x}) = 0 \iff x = -1.76$. Hence, if the support of the unknown distribution of the cost oscillations is $[m,\ M]$, then

$$\beta(m - \zeta) = -1.76 \tag{50}$$

and

$$\beta(M - \zeta) = 6.08 \tag{51}$$

where ζ is the mode of the Gumbel with distribution $\exp(e^{-\beta(x-\zeta)})$. Then, by subtracting (50) from (51), we obtain

$$\beta = \frac{7.84}{M - m}. \tag{52}$$

In our experiments, m is the minimum arc cost of the considered instance while $M := 2\frac{|\mathcal{P}|_{max} * f_{det}}{|\mathcal{I}|}$, where $|\mathcal{P}|_{max} = \max_{(i,j)\in\mathcal{E}} |\mathcal{P}_{ij}|$, is the number of paths considered between each pair of nodes and f_{det} is the value of a deterministic TSP obtained by choosing for each arc that of minimum cost. In this way, we keep M to be proportional to the magnitude of the average cost oscillation in the final solution ($f_{det}/|\mathcal{I}|$), without considering paths with extreme costs.

6.2 Benchmark instances

To better assess the quality and the efficiency of our approximation, we generate two different sets of instances. In the first set (presented in Section 6.2.1) we use well-known distributions for modeling the random cost oscillations, while in the second set (presented in Section 6.2.2) we use a more realistic traffic model.

In both sets, nodes are randomly selected from a database providing the position of 16,862 Italian locations (http://www.math.uwaterloo.ca/tsp/world/it16862.tsp. Last access: December 03, 2018) in terms of Cartesian coordinates, and we assume to have the same number $|\mathcal{P}|$ of available paths for each arc of the network, i.e. $|\mathcal{P}| = |\mathcal{P}_{ij}|, (i, j) \in \mathcal{E}$.

6.2.1 Randomly generated instances

In this set of random instances, the deterministic costs c_{ij}^p are computed as follows $c_{ij}^p := \tau_p * d_{ij}$, where d_{ij} is the Euclidean distance between nodes i and j, and τ_p is randomly sampled in the interval $[1, 3]$. We first create 5 different deterministic instances for each combination of number of nodes ($|\mathcal{I}| = \{50, 100\}$) and number of paths ($|\mathcal{P}| = \{3, 4, 5\}$), i.e. 30 deterministic instances in total. For each deterministic instance, the random cost oscillations Θ_{ij}^{ps}, $(i, j) \in \mathcal{E}, p \in \mathcal{P}_{ij}, s \in \mathcal{S}$ are generated according to 5 different marginal distributions (Gumbel, Normal, Logistic, Laplace, and Uniform). The set of randomly generated instances is therefore composed by 150 instances. We want to highlight since now that the Uniform distribution does not satisfy condition (28), needed to apply our deterministic approximation. Nevertheless, since in real settings it is not always possible to derive a precise knowledge in terms of distribution of the observed scenarios, we want to test the approximation also for distributions that would not fulfill the assumptions of our theory.

To combine the aforementioned marginal distributions into a multivariate one, we use the Normal copula (see Nelsen 2006). This simulates the dependency structure of the random oscillations and maintains the asymptotic independence property. In all cases, the support of the distribution of the random cost oscillation Θ_{ij}^{ps} has been truncated to $[-0.8c_{ij}^p,\ 0.8c_{ij}^p]$ in order to consider significant changes in costs.

6.2.2 Instances based on a realistic traffic model.

In this set of realistic instances, the deterministic costs c_{ij}^p has been assumed to be proportional to the time required to travel from node i to j on path p, thus $c_{ij}^p := c_0 * t_{ij}^p$, where c_0 is a constant value representing the cost per unit of time, and t_{ij}^p is the time required to travel from i to j in normal traffic condition (i.e., without congestion) on path p. For each path $p \in \mathcal{P}_{ij}$, let q_{ij}^p, v_{ij}^p, and l_{ij}^p denote the capacity (i.e., the maximum traffic flow that such path can support), the average speed, and the length of path p, respectively. In order to model a network containing both high capacity paths (main roads or highways) and low capacity ones (secondary roads), q_{ij}^p are randomly generated from a Uniform distribution with support in $\mathcal{Q}_1 = [70, 100]$ for half of the paths and in $\mathcal{Q}_2 = [20, 50]$ for the other half. The average speed v_{ij}^p on any path $p \in \mathcal{P}_{ij}$ is assigned according to the type of the path p. More precisely, we set $v_{ij}^p = 100$ if $q_{ij}^p \in \mathcal{Q}_1$ and $v_{ij}^p = 40$ if $q_{ij}^p \in \mathcal{Q}_2$. The length l_{ij}^p is obtained by uniformly sampling a value in the interval $[d_{ij},\ 3{*}d_{ij}]$. The time t_{ij}^p is then computed as $t_{ij}^p = l_{ij}^p / v_{ij}^p$.

For each deterministic instance, the random costs oscillations Θ_{ij}^{ps} are obtained by using the following formula

$$\Theta_{ij}^{ps} = c_{ij}^p \left(0.15 \left(\frac{Q_{ij}^{ps}(\lambda_{ij})}{q_{ij}^p} \right)^{3+\lambda_{ij}} + \delta_{ij}^{ps} \right) \tag{53}$$

where $Q_{ij}^{ps}(\lambda_{ij})$ is the actual traffic flow on path $p \in \mathcal{P}_{ij}$ under scenario $s \in \mathcal{S}$ and δ_{ij}^{ps} is an additive term generated from a standard Normal distribution truncated in $[-0.3, 0.3]$. δ_{ij}^{ps} models the effect of exogenous events (e.g., weather conditions and road works) that may affect the travel cost except the traffic flow. Note that equation (53) is based on the already presented *Bureau of Public Road* (BPR) model (U.S. Department of Commerce, Bureau of Public Roads 1964), which is widely used and consolidated in transportation engineering. More precisely, the function has been slightly modified by including a positive real parameter λ_{ij} that enables us to modulate the traffic flow on a specific arc (i, j) and therefore to simulate different traffic conditions.

The flows $Q_{ij}^{ps}(\lambda_{ij})$ are computed as follows. First, under each scenario $s \in \mathcal{S}$, the total flow of traffic Q_{ij}^{s} is randomly generated in the interval $[0.3 \sum_{p \in P_{ij}} q_{ij}^{p},\ 0.7 \sum_{p \in P_{ij}} q_{ij}^{p}]$ for low-congested networks, and in $[0.7 \sum_{p \in P_{ij}} q_{ij}^{p},\ \sum_{p \in P_{ij}} q_{ij}^{p}]$ for high-congested networks. Second, the flows $Q_{ij}^{ps}(\lambda_{ij})$ for each path $p \in \mathcal{P}_{ij}$ are then computed as $Q_{ij}^{ps}(\lambda_{ij}) := Q_{ij}^{s} * \pi_{ij}^{p}$, where π_{ij}^{p} denotes the probability of choosing path p among all paths available and can be calculated according to the following Logit model

$$\pi_{ij}^{p} = \frac{\exp(-\lambda_{ij} * (l_{ij}^{p} - l_{ij}^{o}))}{\sum_{p \in \mathcal{P}_{ij}} \exp(-\lambda_{ij} * (l_{ij}^{p} - l_{ij}^{o}))} \tag{54}$$

where l_{ij}^{o} denotes the length of the shortest path p_{ij}^{o} between node i and j. It is worth noting that the costs obtained on the paths by this simulation are necessary dependent because $\sum_{p \in \mathcal{P}_{ij}} Q_{ij}^{ps}(\lambda_{ij}) = Q_{ij}^{s}$. The rationale behind the above formula is the following. Let assume that a user has to make a choice among all paths linking nodes i and j. It is obvious that under normal traffic conditions (no congestion) the user would choose with high probability the shortest path p_{ij}^{o}. Furthermore, the longer a path is the smaller the chance to be selected. Instead, under traffic congestion, the shortest path is surely overused. Then, users try to minimize their travel time by evaluating the possibility to use alternative paths and thus the traffic tends to be redistributed uniformly among all paths (Wardrop principle). These aspects are captured by the Logit model in (54). In fact, when λ_{ij} is close to 0 (high-congested network), the probability of choosing path $p \in \mathcal{P}_{ij}$ tends to $\frac{1}{|\mathcal{P}_{ij}|}$ for all paths. Instead, for large values of λ_{ij} (low-congested network), the probability tends to 0 for all path $p \in \mathcal{P}_{ij} \setminus p_{ij}^{o}$ and to 1 if $p = p_{ij}^{o}$.

Eventually, we have generated a total of 144 instances. More precisely, for each combination of $|\mathcal{I}| = \{50, 100\}$ number of nodes and $|\mathcal{P}| = \{3, 4, 5\}$ number of paths, we have created

- 10 instances (representing high-congested networks) where λ_{ij} is randomly selected in the interval $[0.1, 2]$, $(i, j) \in \mathcal{E}$;
- 10 instances (representing low-congested networks) where λ_{ij} is randomly selected in the interval $[8, 20]$, $(i, j) \in \mathcal{E}$;
- 4 instances (representing a mixed situation showing both congested and not congested paths) where λ_{ij} is randomly selected in the interval $[0.1, 20]$, $(i, j) \in \mathcal{E}$.

6.3 Results and analysis

In order to quantify the performance of the proposed methodology, we run the DA and the DEP (here used as a benchmark) approaches on each generated instance and calculated the percentage gaps $f\%$ and $t\%$ in terms of objective function value of the returned solution and computational time, i.e.

$$f\% := 100 * \frac{f_{DEP} - f_{DA}}{f_{DEP}}$$

where f_{DA} and f_{DEP} are the values of the objective function of the solution computed by using the proposed DA and by solving DEP in (9)–(15) through Cplex within a threshold time of 7200 seconds (2 hours), while

$$t\% := 100 * \frac{t_{DEP} - t_{DA}}{t_{DEP}}$$

where t_{DA} and t_{DEP} are the time required to solve the DA and the DEP, respectively.

We precise that the value of f_{DA} is not obtained directly from the objective function in (49), which only represents an approximation of the overall decision process cost. Instead, a more reasonable evaluation of the real objective function can be obtained, for each instance, through the following steps

1. Optimally solve the model in (49) and derive, for each arc $(i, j) \in \mathcal{E}$, the values y^*_{ij} of the variables y_{ij} in the optimal solution, which represent the first-stage decisions;

2. For each scenario $s \in \mathcal{S}$, solve the DEP (9)–(15) in which the y_{ij} variables are fixed to the values y^*_{ij} found at step 1, calculating the relative objective function f^s_{DA}. Note that, through this variable fixing, the optimization problem actually resorts to simply computing $f^s_{DA} := \sum_{(i,j)\in\mathcal{E}} y^*_{ij} \min_{p\in\mathcal{P}_{ij}} C^{ps}_{ij}$;

3. Finally, compute $f_{DA} := \sum_{s\in\mathcal{S}} \pi^s f^s_{DA}$.

Basically, f_{DA} represents the expected cost that can be obtained by implementing at the first stage the decisions suggested by the deterministic problem derived through our approximation.

Table 1 shows the percentage gaps $f\%$ obtained comparing the DA and the DEP approaches on the 150 instances randomly generated from theoretical distributions. More precisely, each entry reports the average and the standard deviation (in square brackets) of the percentage gaps $f\%$ among the five random instances generated for each number of nodes, number of paths, and type of distribution. The results observed are quite good

Instance		Distribution					
$\lvert\mathcal{I}\rvert$	$\lvert\mathcal{P}\rvert$	Gumbel	Laplace	Logistic	Normal	Uniform	avg:
50	3	0.23 [0.17]	2.12 [2.44]	0.92 [1.08]	1.09 [0.93]	2.14 [2.85]	**1.30** [1.49]
	4	0.58 [0.32]	0.53 [0.57]	0.61 [0.64]	2.00 [2.29]	2.58 [0.93]	**1.26** [0.95]
	5	1.62 [3.19]	0.66 [0.45]	0.31 [0.68]	0.66 [0.75]	1.08 [0.76]	**0.87** [1.17]
	avg:	**0.81** [1.23]	**1.10** [1.16]	**0.62** [0.80]	**1.25** [1.32]	**1.93** [1.52]	**1.14** [1.20]
100	3	1.76 [2.71]	0.69 [0.67]	0.71 [0.37]	2.46 [2.25]	2.02 [1.52]	**1.53** [1.50]
	4	1.26 [0.82]	0.70 [0.49]	1.27 [0.83]	0.62 [0.65]	1.41 [1.16]	**1.05** [0.79]
	5	0.60 [0.69]	2.58 [1.51]	6.11 [9.25]	2.10 [2.22]	1.52 [0.61]	**2.58** [2.86]
	avg:	**1.21** [1.41]	**1.32** [0.89]	**2.70** [3.48]	**1.72** [1.71]	**1.65** [1.10]	**1.72** [1.72]

Table 1: Percentage gaps ($f\%$) obtained for the 150 random generated instances.

in terms of quality. The overall average gaps are 1.14% for all instances with 50 nodes and 1.72% for those with 100 nodes. The standard deviations reported confirm the good stability of the approximation. All the average gaps (and deviations) just slightly increase by increasing the size of the instances (in terms of nodes or paths) for almost all types of distribution. The worse behavior can be found for the Logistic distribution, in particular concerning the largest instances ($|\mathcal{I}| = 100$ and $|\mathcal{P}| = 5$), for which an average gap of about 6% is observed. In all remaining cases, average gaps never exceed 2.6% and are less than 1% in almost the half of cases.

A special attention must be put on the results concerning the Uniform distribution. As already highlighted, even if it does not satisfy assumption (28), we have decided to include these experiments in order to investigate the behavior of our method when the distribution of the oscillations is not known. Also in this case, the observed gaps are very interesting and in line with the other distributions (i.e., 1.93% for $|\mathcal{I}| = 50$ and 1.65% for $|\mathcal{I}| = 100$). This gives to our deterministic approximation an even broader applicability, since it is expected to provide accurate results for a class of distribution even larger than the Gumbel domain of attraction (possibly indicating that assumption (28) is just sufficient but not necessary for the derivation of our results).

The quality of our approximation method has been also tested considering a more realistic traffic model, which totally disregards any assumption on the resulting empirical distribution of the costs. The percentage gaps $f\%$ obtained comparing the DA and the DEP approaches on the 144 realistic instances are shown in Table 2. Again, each entry reports the average and the standard deviation (in square brackets) of the percentage gaps $f\%$ among the random instances generated for each number of nodes, number of paths, and type of traffic congestion. Also in this case, the results obtained are still good, also considering the complexity of the underlying problem. The approximation provides on average a solution differing from the one of the DEP by less than 4.6% for all type of networks (with reasonable standard deviations). Furthermore, such gaps seem not increasing as the number of nodes or paths increases, demonstrating again good stability and scalability of the approach. On the contrary, the approximation works better when the number of nodes and the number of possible alternatives per arc (paths) are higher.

Instance		Type of traffic			
$\|\mathcal{I}\|$	$\|\mathcal{P}\|$	Low congestion	Mixed situation	High congestion	avg:
50	3	2.99 [1.61]	7.36 [11.02]	3.98 [2.81]	**4.78** [5.15]
	4	3.33 [1.78]	3.87 [1.16]	3.92 [2.03]	**3.71** [1.65]
	5	5.74 [2.13]	2.35 [2.67]	5.58 [2.79]	**4.56** [2.53]
	avg:	**4.02** [1.84]	**4.53** [4.95]	**4.49** [2.54]	**4.35** [3.11]
100	3	2.70 [1.41]	2.16 [0.80]	3.91 [1.91]	**2.92** [1.37]
	4	3.99 [1.43]	3.46 [1.32]	5.02 [1.18]	**4.16** [1.31]
	5	4.64 [2.52]	3.07 [1.42]	4.65 [2.40]	**4.12** [2.11]
	avg:	**3.78** [1.78]	**2.90** [1.18]	**4.53** [1.83]	**3.74** [1.60]

Table 2: Percentage gaps ($f\%$) obtained for the 144 realistic instances.

In Table 3, we have summarized the computational times in seconds observed for the two solution approaches (DA and DEP) on both random generated and realistic instances. In both cases, we do not show detailed results per type of distribution or type of network because the differences are not sensible, and mostly depend on the number of nodes. In all the experiments, the time required to get the solution of DA is definitely negligible with respect to the time employed to perform the solution of DEP. DEP needs, on average, about 800 seconds to solve instances with 50 nodes and approaches the threshold time of 2 hours for the instances with 100 nodes. Instead, DA needs on average less than 7 and less than 100 seconds to solve instances with 50 and 100 nodes, respectively. We can also observe that, on average, realistic instances are solved in less time by DA with respect to random ones. Just about 2 and 30 seconds are needed for solving $|\mathcal{I}| = 50$ and 100 instances, respectively.

Instance		Random instances		Realistic instances	
$\|\mathcal{I}\|$	$\|\mathcal{P}\|$	$t_{DA}(s)$	$t_{DEP}(s)$	$t_{DA}(s)$	$t_{DEP}(s)$
50	3	7.0	714.7	1.9	722.1
	4	5.6	1037.0	1.8	437.4
	5	6.0	655.8	2.1	830.0
	avg:	**6.2**	**802.5**	**1.9**	**663.2**
100	3	97.7	6791.1	33.5	6805.1
	4	86.9	5930.6	29.2	6670.0
	5	104.7	7006.2	24.8	7162.9
	avg:	**96.4**	**6576.0**	**29.1**	**6879.3**

Table 3: Computational times of the two approaches for all the generated instances.

Finally, in Tables 4 and 5 we summarize how good is the compromise offered by DA for the two main sets of instances, in terms of efficiency and effectiveness. More precisely, we compare the loss in effectiveness $f\%$ and the gain in efficiency $t\%$ when using the proposed DA with respect to DEP. On average, by sacrificing from 1% to 4% of the solution quality, the approximation allows to gain 2 orders of magnitude in efficiency.

Distribution	$f\%$	$t\%$
Gumbel	1.01	97.93
Laplace	1.21	98.69
Logistic	1.66	96.98
Normal	1.49	98.10
Uniform	1.79	98.46
avg:	**1.43**	**98.03**

Table 4: Loss in effectiveness vs. gain in efficiency of DA with respect to DEP for the random instances.

7 Conclusions

In this paper, we have studied the stochastic multi-path Traveling Salesman Problem with dependent random travel costs ($smpTSP_{dc}$). We have shown that, under a mild assumption on the distribution of the random cost oscillations and if such oscillations are just asymptotically independent, a deterministic approximation of the problem can be derived by using the theory of extreme values. We have also shown that the asymptotically independence assumption on the travel costs is not too restrictive in real network

Traffic situation	$f\%$	$t\%$
Low congestion	4.51	99.60
Mixed situation	3.71	99.80
High Congestion	3.90	99.52
avg:	**4.12**	**99.60**

Table 5: Loss in effectiveness vs. gain in efficiency of DA with respect to DEP for realistic instances.

applications. On the contrary, it allows to deal with realistic traffic models such as the well-known BPR function. Finally, we have tested the behavior of the proposed methodology by means of extensive computational experiments on random generated as well as realistic instances with up to 100 nodes and 5 possible different paths per arc. The deterministic approximation is definitely able to solve the problem with very good compromise between quality of the solution and overall efficiency with respect to standard equivalent Stochastic Programming approaches and state-of-the-art solvers. On average, the deterministic approximation can find in less than 100 seconds solutions with 1-4% of gap with respect to the optimal ones, which need instead hours to be found. We are confident that such percentages could still persist (or even improve) for instances with larger number of nodes, paths, and considered scenarios.

Some future research can be outlined. First, encouraged by the very good results obtained even in those cases where the theoretical assumptions for the derivation of our approximation do not hold, we want to further investigate possible relaxations of such assumptions. Moreover, we want to concentrate on finding a way to calibrate the β parameter which better exploits the instance features. Finally, a time-dependent version of the problem can be studied and approximated through other very recent developments on random utility choice models (Tadei, Perboli, and Manerba 2019).

References

Beraldi P, Bruni ME, Manerba D, Mansini R, 2017 *A stochastic programming approach for the traveling purchaser problem. IMA Journal of Management Mathematics* 28(1):41–63.

Campbell AM, Gendreau M, Thomas BW, 2011 *The orienteering problem with stochastic travel and service times. Annals of Operations Research* 186(1):61–81.

Campbell AM, Thomas BW, 2008 *Probabilistic traveling salesman problem with deadlines. Transportation Science* 42(1):1–21.

Carraway R, Morin T, Moskowit H, 1989 *Generalized dynamic programming for stochastic combinatorial optimization. Operations Research* 5(37):819–829.

Chen BY, Lam W, Sumalee A, Li Q, Lam Tam M, 2014 *Reliable shortest path problems in stochastic time-dependent networks. Journal of Intelligent Transportation Systems* 18:177–189.

Eiger A, Mirchandani PB, Soroush H, 1985 *Path preferences and optimal paths in probabilistic networks. Transportation Science* 19(1):75–84.

Fan Y, Kalaba R, Moore J, 2005 *Shortest paths in stochastic networks with correlated link costs. Computers & Mathematics with Applications* 49(9):1549–1564.

Galambos J, 1978 *The Asymptotic Theory of Extreme Order Statistics* (John Wiley, New York).

Gutin G, Punnen AP, eds., 2002 *The traveling salesman problem and its variations.* Combinatorial Optimization (Dordrecht, London: Kluwer Academic).

Hansen W, 1959 *How accessibility shapes land use. Journal of the American Institute of Planners* 25:73–76.

Huang Z, Zheng QP, Pasiliao E, Boginski V, Zhang T, 2018 *A cutting plane method for risk-constrained traveling salesman problem with random arc costs. Journal of Global Optimization* URL `http://dx.doi.org/10.1007/s10898-018-0708-0`.

Kao E, 1978 *A preference order dynamic program for a stochastic traveling salesman problem. Operations Research* 6(26):1033–1045.

Kaut M, Vladimirou H, Wallace S, Zenios S, 2007 *Stability analysis of portfolio management with conditional value-at-risk. Quantitative Finance* 7:397–409.

Kenyon AS, Morton DP, 2003 *Stochastic vehicle routing with random travel times. Transportation Science* 37(1):69–82.

Kirkpatrick S, Toulouse G, 1985 *Configuration space analysis of traveling salesman problem. J. Phys. France* 46:1277–1292.

Lawler E, Lenstra JK, Rinnooy Kan A, Shmoys D, 1985 *Traveling salesman problem: a guided tour of combinatorial optimization* (Wiley, Chichester).

Letchford AN, Nasiri SD, 2015 *The steiner traveling salesman problem with correlated costs. European Journal of Operational Research* 245(1):62–69.

Maggioni F, Perboli G, Tadei R, 2014 *The multi-path traveling salesman problem with stochastic travel costs: Building realistic instances for city logistics applications. Transportation Research Procedia* 3:528–536, 17th Meeting of the EURO Working Group on Transportation, July 2-4, 2014. Sevilla (Spain).

Manerba D, Mansini R, Riera-Ledesma J, 2017 *The traveling purchaser problem and its variants. European Journal of Operational Research* 259(1):1–18.

Mezard M, Parisi G, 1986 *A replica analysis of the traveling salesman problem. J. Phys. France* 47:1285–1296.

Nelsen RB, 2006 *An Introduction to Copulas*. Springer series in Statistics (Springer-Verlag New York).

Osborne MJ, Rubinstein A, 1994 *A course in game theory* (MIT press).

Perboli G, Tadei R, Baldi M, 2012 *The stochastic generalized bin packing problem. Discrete Applied Mathematics* 160:1291–1297.

Perboli G, Tadei R, Gobbato L, 2014 *The multi-handler knapsack problem under uncertainty. European Journal of Operational Research* 236(3):1000–1007.

Reinelt G, 1994 *The traveling salesman: computational solutions for TSP applications*, volume 840 (Springer, Berlin).

Samaranayake S, Blandin S, Bayen A, 2012 *A tractable class of algorithms for reliable routing in stochastic networks. Transportation Research Part C: Emerging Technologies* 20(1):199–217.

Sniedovich M, 1981 *Analysis of a preference order traveling salesman problem. Operations Research* 6(29):1234–1237.

Tadei R, Perboli G, Manerba D, 2018 *A recent approach to derive the multinomial logit model for choice probability. P. Daniele and L. Scrimali (eds.), New Trends in Emerging Complex Real Life Problems, AIRO Springer Series 1 – ODS2018, Sept 10-13, 2018. Taormina (Italy)*.

Tadei R, Perboli G, Manerba D, 2019 *The multi-stage dynamic stochastic decision process with unknown distribution of the random utilities. Optimization Letters (Submitted)* .

Tadei R, Perboli G, Perfetti F, 2014 *The multi-path traveling salesman problem with stochastic travel costs. EURO Journal on Transportation and Logistics* 6:3–23.

Tadei R, Perboli G, Ricciardi N, Baldi MM, 2012 *The capacitated transshipment location problem with stochastic handling costs at the facilities. International Transactions in Operational Research* 19(6):789–807.

Toriello A, William B, Poremba HM, 2013 *A dynamic traveling salesman problem with stochastic arc costs. Operations Research* 62(5):1107–1125.

Toth P, Vigo D, 2014 *Vehicle Routing: Problems, Methods, and Applications - Second Ed.* MOS-SIAM Series on Optimization (Philadelphia, PA: Society for Industrial and Applied Mathematics).

US Department of Commerce, Bureau of Public Roads, 1964 *Traffic assignment manual* (Washington, U.S. Dept. of Commerce, Bureau of Public Roads, Office of Planning, Urban Planning Division).

Vansteenwegen P, Souffriau W, Oudheusden DV, 2011 *The orienteering problem: A survey. European Journal of Operational Research* 209(1):1–10.

Wallace S, Ziemba W, 2005 *Applications of Stochastic Programming.* MPS-SIAM Series on Optimization (Society for Industrial and Applied Mathematics).

Wardrop JG, 1952 *Road paper. Some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers* 1(3):325–362, URL `http://dx.doi.org/10.1680/ipeds.1952.11259`.

Wardrop JG, Whitehead JI, 1952 *Correspondence. Some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers* 1(5):767–768, URL `http://dx.doi.org/10.1680/ipeds.1952.11362`.

Wästlund J, 2010 *The mean field traveling salesman and related problems. Acta Mathematica* 204(1):91–150.